JOURNAL OF APPROXIMATION THEORY **89**, 1–25 (1997)
ARTICLE NO. AT963035

Characterization of Best Algebraic Approximation by an Algebraic Modulus of Smoothness

Michael Felten

Lehrstuhl VIII für Mathematik, Universität Dortmund, D-44221 Dortmund, Germany
E-mail: felten@bernstein.mathematik.uni-dortmund.de

Communicated by Z. Ditzian

Received December 14, 1994; accepted in revised form April 10, 1996

The author introduced in an earlier paper a modulus of smoothness for non-periodic functions based on an algebraic addition $\oplus$ defined on $[-1, 1]$. In this paper the Steklov functions and an equivalent K-functional are given. Moreover, a characterization of best approximation by algebraic polynomials and the equivalence between the algebraic modulus and the Butzer–Stens modulus, introduced by the Chebyshev translation method, are shown.

1. INTRODUCTION

Nikolskiĭ showed in 1946 that the approximation of continuous functions defined on a finite interval, i.e., $[-1, 1]$ by algebraic polynomials, is better near the endpoints ± 1 than inside of $[-1, 1]$ and therefore is not uniformly good over the whole interval. The endpoints ± 1 therefore play an exceptional role. It is well known that the rate of algebraic approximation to a function $f \in C[-1, 1]$ cannot be characterized in terms of the classical modulus

$$\omega(f; \delta) = \sup_{|h| \leqslant \delta} \sup_{x,\, x+h \in [-1, 1]} |f(x+h) - f(x)|. \tag{1.1}$$

This is basically due to the fact that the ordinary translation $x + h$ does not cover the situation at the endpoints.

In [7] the translation $x + h$ of the modulus (1.1) was replaced by the weighted addition

$$x \oplus h := x \sqrt{1-h^2} + \sqrt{1-x^2}\, h, \qquad x, h \in [-1, 1], \tag{1.2}$$

0021-9045/97 $25.00

which is an inner operation on the unit interval $E := [-1, 1]$, i.e., $\oplus: E \times E \to E$. The forward difference operator for functions $f: [-1, 1] \to \mathbb{R}$ can then be defined as

$$(\Delta_h f)(x) := f(x \oplus h) - f(x) \qquad \text{for all} \quad x \in [-1, 1]$$

and the difference operator of rth order is given by composition $\Delta_h^r = \Delta_h \circ \cdots \circ \Delta_h$ (r times).

The concept of measuring smoothness via the difference operator Δ_h^r has been introduced in [7] for the space $C[-1, 1]$ and for the weighted space $\mathrm{L}_\varphi^p[-1, 1]$, $p \in [1, \infty)$, whereby the former denotes the set of all continuous functions $f: [-1, 1] \to \mathbb{R}$ endowed with the norm

$$\|f\|_\infty := \sup_{x \in [-1, 1]} |f(x)|$$

and the latter is the set of all measurable functions $f: [-1, 1] \to \mathbb{R}$ for which the norm

$$\|f\|_{p, \varphi} := \left(\int_{-1}^{1} |f(x)|^p \frac{dx}{\varphi(x)} \right)^{1/p}, \qquad \varphi(x) = \sqrt{1 - x^2},$$

is finite. For the sake of abbreviation let us set

$$X = \begin{cases} \mathrm{C}[-1, 1], & p = \infty \\ \mathrm{L}_\varphi^p[-1, 1], & p \in [1, \infty) \end{cases} \quad \text{and} \quad \|\bullet\|_X = \begin{cases} \|\bullet\|_\infty, & p = \infty \\ \|\bullet\|_{p, \varphi}, & p \in [1, \infty) \end{cases}.$$

It was shown that the modulus of smoothness

$$\mathrm{w}_\varphi^r(f; \delta)_X := \sup_{|h| \leqslant \delta} \|\Delta_h^r f\|_X, \qquad \delta \in [0, 1], \tag{1.3}$$

is well defined for all $f \in X$.

One of the goals of this paper will be to prove that the Jackson and Bernstein assertions with respect to the approximation by algebraic polynomials are equivalent, i.e., the characterization

$$\mathrm{E}_n(f)_X = O(n^{-\alpha}) \quad (n \to \infty) \Leftrightarrow \mathrm{w}_\varphi^r(f; \delta)_X = O(\delta^\alpha) \qquad (\delta \to 0)$$

holds true for all $f \in X$ and α with $0 < \alpha < r$, where $\mathrm{E}_n(f)_X = \inf_{p_n \in \mathbb{P}_n} \|f - p_n\|_X$ and $\mathbb{P}_n$ denotes the set of all algebraic polynomials of degree n.

Let us very briefly touch on recent developments. It was A. F. Timan in 1951 [11] who first proved that $\omega(f; \delta) = O(\delta^\alpha)$ $(0 < \alpha < 1, f \in \mathrm{C}[-1, 1])$ is equivalent to the fact that there exists a sequence $\{p_n\}$ of algebraic polynomials of degree n which satisfies

$$\|(\varphi + n^{-1})^{-\alpha} (f - p_n)\|_\infty = O(n^{-\alpha}) \qquad \text{with} \quad \varphi(x) = \sqrt{1 - x^2}.$$

Unfortunately, the corresponding result for weighted and unweighted L^p approximation does not hold true, a fact which was pointed out by V. P. Motorniĭ (1971) [10] and R. DeVore [3]. Another approach consisted in manipulating the modulus of smoothness in order to characterize the rate of algebraic approximation. This was solved in different ways and in different spaces by P. L. Butzer and R. L. Stens [1, 2], K. G. Ivanov [8, 9], and Z. Ditzian and V. Totik [5].

The translation defined in (1.2) is similar to the translation $x+\sqrt{1-x^2}\,h$ introduced by Ditzian and Totik [5]. The increment $\sqrt{1-x^2}\,h$ of the Ditzian–Totik translation varies together with x and becomes smaller if x is near the endpoints. This also holds true for $x \oplus h$ if h is small. But in contrast to $x+\sqrt{1-x^2}\,h$ the double weighted translation (1.2) always remains inside of the unit interval $[-1, 1]$.

Although the difference between $x \oplus h$ and the Ditzian–Totik translation is only $O(h^2)$ for $h \to 0$, the modulus (1.3) behaves differently from the Ditzian–Totik modulus. It was found that (1.3) is not well-defined for unweighted L^p spaces on $[-1, 1]$ ($p \in [1, \infty)$), whereas the Ditzian–Totik modulus is. This aspect will be discussed in more detail in Section 5.

Our approach is based on the elementary properties of the algebraic addition and of the modulus which have been investigated in [7]. The main results of [7] will be reviewed in Section 2.

Whereas the paper [7] is mainly concerned with the introduction of the w_φ^r modulus, this paper deals with applications. In Section 3 Steklov functions are introduced via the algebraic addition $\oplus$. It will be seen that these Steklov functions possess properties similar to those of the classical Steklov functions. As the differential operator D we use $Df := \varphi \cdot f'$ for $f \in \mathrm{AC}[-1, 1]$. The power D^r is defined by composition $D \circ \cdots \circ D$ r times.

Section 4 investigates a suitable K-functional which is defined by means of the differential operator D^r. One of the aims of this paper is to show that K-functional and modulus w_φ^r are equivalent. We are able to give values to the constants in the inequalities.

Finally, it will be shown that the algebraic modulus w_φ^{2r} of even order is equivalent to the Butzer–Stens modulus ω_T^r, introduced by the Chebyshev translation method [1, 2].

2. PROPERTIES OF w_φ^r MODULI OF SMOOTHNESS

Before we can work with the modulus, we have to recall the basic properties of the addition $\oplus$ (for more details see [7]). $\oplus$ is commutative, 0 is the neutral element, and $-a$ is the inverse of a. So it is convenient to define the subtraction by $a \ominus b := a \oplus (-b) = a\sqrt{1-b^2} - \sqrt{1-a^2}\,b$ for all $a, b \in [-1, 1]$. ($[-1, 1]$, $\oplus$) is not a group, because the associative law is missing. But the associative law is fulfilled on certain subintervals of the unit interval $E := [-1, 1]$.

DEFINITION 2.1. Let us denote as E_h, $h \in E = [-1, 1]$, the following subintervals

$$E_h := [-1, 1]_h := \begin{cases} [-1, 1 \ominus h] & \text{for} \quad h \geqslant 0 \\ [-1 \oplus h, 1] & \text{for} \quad h < 0 \end{cases}.$$

Then the associative law $(a \oplus h) \oplus b = a \oplus (h \oplus b)$ holds true for every $a, b \in \mathrm{E}_h$. Moreover, there are three remaining cases, the only difference is that a minus sign appears at different positions:

$$\begin{aligned} (a \oplus h) \oplus b &= -a \oplus (h \oplus b) && \text{for} \quad a \in E_h, \quad b \in E \backslash E_h, \\ (a \oplus h) \oplus b &= a \oplus (h \ominus b) && \text{for} \quad a \in E \backslash E_h, \quad b \in E_{-h}, \\ (a \oplus h) \oplus b &= -a \oplus (h \ominus b) && \text{for} \quad a \in E \backslash E_h, \quad b \in E \backslash E_{-h}. \end{aligned}$$

In [7] the associative law for n summands

$$(\cdots((a_1 \oplus a_2) \oplus a_3) \oplus \cdots) \oplus a_n \tag{2.1}$$

was investigated for $n \geqslant 3$. It was shown that the parentheses in (2.1) may be omitted without ambiguity if

$$a_1, ..., a_n \in \left[-\frac{1}{(n-2)\sqrt{2}}, \frac{1}{(n-2)\sqrt{2}} \right]. \tag{2.2}$$

It was pointed out in [7] that the difference operator $(\Delta_h^r f)(x)$ for $r \geqslant 2$ manifests a completely different behavior for x near the endpoints ± 1 than the corresponding classical difference operator.

THEOREM 2.2. *Let* $r \in \mathbb{N}$, f: $[-1, 1] \to \mathbb{R}$ *and* $h \in E := [-1, 1]$. *Then the* r*th difference operator satisfies*

$$(\Delta_h^r f)(x) = (-2)^{r-1} \cdot (\Delta_h f)(x)$$

for each $x \in E \backslash E_h$.

Observe that

$$E \backslash E_h = \begin{cases} [\sqrt{1-h^2}, 1], & h \geqslant 0 \\ [-1, -\sqrt{1-h^2}], & h < 0 \end{cases}$$

and these two intervals consist of points which lie near ± 1. Therefore the order of the difference operator is reduced to one if x is near the endpoints. This is due to the fact that the associative law is missing around the boundaries. In contrast to the classical moduli of higher order it turns out that in $\mathrm{w}_{\varphi}^{r}(f;\delta)_X$ the order varies near the endpoints of $[-1,1]$. This shows the difference between Δ_h^r and the corresponding difference operator of the periodic case. Therefore a reduction to the periodic situation by some cosine substitution is not possible.

It should be taken into consideration that the starting point in [7] was the Ditzian–Totik translation $x+\sqrt{1-x^2}\cdot h$, where the increment varies together with x and becomes smaller if x is near the endpoints ± 1. The same holds true for the algebraic addition. But now the order of the difference operator, too, varies together with x and becomes smaller if x is near the endpoints. This is altogether in keeping with the fact that the algebraic approximation near the endpoints is better than inside the interval $[-1,1]$.

In the next section the following properties of the algebraic translation $x\mapsto x\oplus h$ will be of interest for Steklov functions.

THEOREM 2.3. *Let* $\tau_h\colon[-1,1]\to[-1,1]$ *defined as* $\tau_h(x):=x\oplus h$ *for* $h\in[-1,1]$.

(i) *For the restrictions* τ_h *on* E_h *and* $E\backslash E_h$ *respectively, the following properties are satisfied*:

$$\tau_h\colon E_h\to E_{-h},\qquad \tau_h^{-1}=\tau_{-h},$$
$$\tau_h\colon E\backslash E_h\to E\backslash E_h,\qquad \tau_h^{-1}=\tau_h.$$

τ_h *is on the above subintervals bijective and has the mentioned inverses.*

(ii) τ_h *is on* $(-1,1)$ *continuously differentiable and*

$$\tau_h'(x)=\begin{cases}\dfrac{1\oplus\tau_h(x)}{1\oplus x} & \text{for}\quad x\in E_h\\[2ex] -\dfrac{1\oplus\tau_h(x)}{1\oplus x} & \text{for}\quad x\in E\backslash E_h\end{cases}.$$

By application of Theorem 2.3 (i) one can show that the inequality

$$\|f(\bullet\oplus h)\|_X\leqslant 2^{1/p}\,\|f\|_X \tag{2.3}$$

holds true. As a consequence we get the following two estimations:

$$\begin{aligned}\mathrm{w}_{\varphi}^{r}(f;\delta)_X&\leqslant(2\cdot 2^{1/p})^r\cdot\|f\|_X,\\ \mathrm{w}_{\varphi}^{r+s}(f;\delta)_X&\leqslant(2\cdot 2^{1/p})^s\cdot\mathrm{w}_{\varphi}^{r}(f;\delta)_X.\end{aligned}\tag{2.4}$$

For later use we need an algebraic multiplication. In [7] $\odot\colon \mathbb{N}_0 \times E \to E$ was defined by

$$n \odot x := \underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}, \qquad x \in E,$$

for every $n \in \mathbb{N}$ and $n \odot x := 0$ for $n = 0$. It was proved that

$$|n \odot x| \leqslant n \cdot |x| \tag{2.5}$$

holds true.

The next two theorems will be needed for Steklov functions. The proofs can be found in [7].

THEOREM 2.4. *Let $n, r \in \mathbb{N}$, $r \geqslant 2$, and $\delta^* := 1/((nr-1)\sqrt{2})$. Then*

$$(\Delta^r_{n \odot h} f)(x) = \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} (\Delta^r_h f)(x \oplus (k_1 \odot h) \oplus \cdots \oplus (k_r \odot h))$$

holds true for every $x \in [-1, \delta^*]$, $h \in [0, \delta^*]$ and every $x \in [-\delta^*, 1]$, $h \in [-\delta^*, 0]$.

THEOREM 2.5. *Let $f \in \mathrm{L}^1_\varphi[-1, 1]$, $n \in \mathbb{N}$ and $x \in [-1/(n\sqrt{2}), 1/(n\sqrt{2})]$. The multiplication $\odot$ posseses the following properties*

(i) $\displaystyle (n \odot x)' = n \cdot \frac{1 \oplus (n \odot x)}{1 \oplus x},$

(ii) $\displaystyle \int_0^{n \odot x} f(t) \frac{dt}{\sqrt{1-t^2}} = n \cdot \int_0^x f(n \odot t) \frac{dt}{\sqrt{1-t^2}},$

(iii) $\arcsin(n \odot x) = n \cdot \arcsin x.$

3. STEKLOV MEANS

Steklov means provide a way to approximate a given (not necessarily continuous) function by smooth functions. Here Steklov functions are defined by means of the addition $\oplus$.

DEFINITION 3.1. Let $f \in \mathrm{L}^1_\varphi[-1, 1]$ and $h \in [-1, 1] \setminus \{0\}$. The Steklov function $f_h\colon [-1, 1] \to \mathbb{R}$ of first order is given by

$$f_h(x) := \frac{1}{\arcsin h} \int_0^h f(x \oplus t) \frac{dt}{\sqrt{1-t^2}} \qquad \text{for} \quad x \in [-1, 1]. \tag{3.1}$$

Steklov functions of higher order are defined recursively by

$$f_{h,r+1} := (f_{h,r})_h \qquad \text{for} \quad r \in \mathbb{N}, \quad f_{h,1} := f_h.$$

Let us recall that $C[-1,1] \subset L^p_\varphi[-1,1] \subset L^1_\varphi[-1,1]$ and hence the Steklov functions $f_{h,r}$ are defined for the space X. Moreover, if f is constant ($f = c$) then the Steklov function is also constant ($f_{h,r} = c$).

Let D be the differential operator $Df := \varphi \cdot f'$ for $f \in AC[-1,1]$. The power D^r is defined by the composition $D \circ \cdots \circ D$ r times. In addition we define the *Sobolev space* with respect to D as

$$W^r_X[-1,1] := \{f \mid f, Df, ..., D^{r-1}f \in AC[-1,1], D^r f \in X\}, \qquad r \in \mathbb{N}.$$

Throughout, the abbreviation "(a.e.)" means that an assertion holds for almost all x if $X = L^p_\varphi[-1,1]$, and for all x if $X = C[-1,1]$.

THEOREM 3.2. *Let $f \in L^1_\varphi[-1,1]$ and $h \in [-1,1] \backslash \{0\}$. Then $f_h \in AC[-1,1]$ and*

$$(Df_h)(x) = \frac{(\Delta_h f)(x)}{\arcsin h} \qquad \textit{for} \quad x \in [-1,1] \quad (\textit{a.e.}).$$

Proof. Let $F: [-1,1] \to \mathbb{R}$ be the following absolutely continuous function

$$F(x) := \int_0^x f(t) \frac{dt}{1 \oplus t}, \qquad x \in [-1,1] =: E.$$

F has the derivative

$$F'(x) = \frac{f(x)}{1 \oplus x}, \qquad x \in E \quad \text{(a.e.)}. \tag{3.2}$$

Let us fix $h > 0$ and $x \in E$. Now we shall derive a representation for the Steklov function f_h by means of F. It is useful to consider two cases.

Case $x \in E_h$. In this case an easy calculation yields the inclusion $[0, h] \subset E_x$. This implies by substitution $t \mapsto t \ominus x$ and Theorem 2.3 that

$$\arcsin h\, f_h(x) = \int_0^h f(x \oplus t) \frac{dt}{1 \oplus t} = \int_x^{x \oplus h} f(t) \frac{dt}{\sqrt{1-t^2}} = F(x \oplus h) - F(x)$$

and therefore from (3.2)

$$\arcsin h\, f_h'(x) = F'(x \oplus h) \frac{1 \oplus (x \oplus h)}{1 \oplus x} - F'(x) = \frac{f(x \oplus h) - f(x)}{1 \oplus x}.$$

Case $x \in E \backslash E_h$. Let us split up the interval $[0, h]$ of integration into the following subintervals

$$[0, h] = J_1 \cup J_2, \qquad J_1 \subset E_x, \qquad J_2 \subset E \backslash E_x.$$

In this case a short computation yields $J_1 = [0, 1 \ominus x]$ and $J_2 = (1 \ominus x, h]$. Since $J_1 \subset E_x$ and $J_2 \subset E \backslash E_x$ we get by substitution $t \mapsto t \oplus x$ and $t \mapsto t \ominus x$

$$\begin{aligned}\arcsin h\, f_h(x) &= \int_0^{1 \ominus x} f(x \oplus t) \frac{dt}{1 \ominus t} + \int_{1 \ominus x}^{h} f(x \oplus t) \frac{dt}{1 \ominus t} \\ &= \int_x^1 f(t) \frac{dt}{\sqrt{1-t^2}} + \int_{x \oplus h}^{1} f(t) \frac{dt}{\sqrt{1-t^2}} \\ &= 2F(1) - F(x) - F(x \oplus h)\end{aligned}$$

for all $x \in E \backslash E_h$. It follows from Theorem 2.3 and (3.2) that

$$\begin{aligned}\arcsin h\, f_h'(x) &= -F'(x) - F'(x \oplus h) \cdot \left(-\frac{1 \oplus (x \oplus h)}{1 \oplus x} \right) \\ &= \frac{f(x \oplus h) - f(x)}{1 \oplus x},\end{aligned}$$

which verifies the representation of the assertion. Observing that f_h is absolutely continuous on E_h and $E \backslash E_h$ and f_h is continuous on the union set $E_h \cup E \backslash E_h = [-1, 1]$, f_h is absolutely continuous on $[-1, 1]$.

Now let $h < 0$. We can trace this case back to $h > 0$ by setting $g(x) := f(-x)$ for $x \in [-1, 1]$. Because

$$\begin{aligned}f_h(x) &= \frac{-1}{\arcsin h} \int_0^{-h} f(x \ominus t) \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{\arcsin(-h)} \int_0^{-h} g((-x) \oplus t) \frac{dt}{\sqrt{1-t^2}} = g_{-h}(-x),\end{aligned}$$

f_h is in $\mathrm{AC}[-1, 1]$ with the following derivative

$$\begin{aligned}f_h'(x) &= -g'_{-h}(-x) = -\frac{g((-x) \oplus (-h)) - g(-x)}{\arcsin(-h) \cdot \varphi(-x)} \\ &= \frac{f(x \oplus h) - f(x)}{\arcsin h \cdot \varphi(x)}. \quad \blacksquare\end{aligned}$$

The result for higher derivatives is given in

THEOREM 3.3. *Let* $f \in X$, $r \in \mathbb{N}$, $r \geqslant 2$ *and* $\delta_r := 1/((r-1)\sqrt{2})$. *Furthermore, let* $h \in [-\delta_r, \delta_r] \backslash \{0\}$ *and*

$$[a, b] := \begin{cases} [-1, \delta_r], & h > 0 \\ [-\delta_r, 1], & h < 0 \end{cases}.$$

Then $f_{h,r} \in W^r_X[a, b]$ *and*

$$(D^r f_{h,r})(x) = \frac{(\Delta^r_h f)(x)}{(\arcsin h)^r} \qquad \text{for} \quad x \in [a, b] \quad (\text{a.e.}).$$

Proof. It is useful to introduce the piecewise constant function

$$\mu_h \colon [-1, 1] \to \{\pm 1\}, \qquad \mu_h(x) := \begin{cases} +1, & x \in E_h \\ -1, & x \in E \backslash E_h \end{cases}. \tag{3.3}$$

Later on we show $[a, b] \subset E_h$. Because of Theorem 2.3(ii) we have

$$\frac{d}{dx}(x \oplus h) = \mu_h(x) \cdot \frac{1 \oplus (x \oplus h)}{1 \oplus x} = \mu_h(x) \cdot \frac{\varphi(x \oplus h)}{\varphi(x)}$$

and

$$D(f(\bullet \oplus h)) = \varphi \cdot f'(\bullet \oplus h) \cdot \mu_h \cdot \frac{\varphi(\bullet \oplus h)}{\varphi} = \mu_h \cdot (Df)(\bullet \oplus h)$$

or more generally

$$(D^j(f(\bullet \oplus h)))(x) = (\mu_h(x))^j \cdot (D^j f)(x \oplus h), \qquad j \in \mathbb{N}. \tag{3.4}$$

Then by (3.4) and Theorem 3.2 we get for $j \in \mathbb{N}$

$$\begin{aligned} &(D^{j+1} f_{h,j+1})(x) \\ &\quad = (D^j(D(f_{h,j})_h))(x) \\ &\quad = (\arcsin h)^{-1} \, (D^j(f_{h,j}(\bullet \oplus h) - f_{h,j}(\bullet)))(x) \\ &\quad = (\arcsin h)^{-1} \, (\mu^j_h(x)(D^j f_{h,j})(x \oplus h) - (D^j f_{h,j})(x)) \qquad (\text{a.e.}) \end{aligned}$$

and

$$(D^{j+1} f_{h,j+1})(x) = (\arcsin h)^{-1} ((D^j f_{h,j})(x \oplus h) - (D^j f_{h,j})(x)),$$
$$x \in E_h \quad \text{(a.e.)}. \tag{3.5}$$

Now we are going to prove the following assertion: For $r \geqslant 2$, $h \in [-1, 1] \backslash \{0\}$ and $x \in [-1, 1]$ with

$$(\cdots(x \oplus \underbrace{h) \oplus \cdots) \oplus h}_{k \text{ summands } h} \in E_h \qquad \text{for} \quad k = 0, ..., r-2 \tag{3.6}$$

the following equation holds true:

$$(D^r f_{h,r})(x) = \frac{(\Delta_h^r f)(x)}{(\arcsin h)^r} \qquad \text{(a.e.)}. \tag{3.7}$$

We prove this by induction with respect to r. Note that (3.6), $k=0$, implies $x \in E_h$ and therefore $\mu_h(x) = 1$.

For $r=2$ by using Eq. (3.5) ($j=1$) and Theorem 3.2 we have

$$\begin{aligned}(D^2 f_{h,2})(x) &= (\arcsin h)^{-1} ((D f_h)(x \oplus h) - (D f_h)(x)) \\ &= (\arcsin h)^{-2} ((\Delta_h f)(x \oplus h) - (\Delta_h f)(x)) \\ &= (\arcsin h)^{-2} (\Delta_h^2 f)(x) \qquad \text{(a.e.)}.\end{aligned}$$

Let the statement hold true for a given number r. We have to show for $r+1$ that condition (3.6) implies Eq. (3.7). Therefore let us assume that (3.6) is fulfilled for $r+1$, i.e. (3.6) holds true for $k = 0, ..., r-1 = (r+1)-2$. This means that Eq. (3.7) also holds true for $x \oplus h$ instead of x (because of the assumption of the induction). Using (3.5) (note $\mu_h(x) = 1, j = r$) gives

$$\begin{aligned}(D^{r+1} f_{h,r+1})(x) &= (\arcsin h)^{-(r+1)} ((\Delta_h^r f)(x \oplus h) - (\Delta_h^r f)(x)) \\ &= (\arcsin h)^{-(r+1)} (\Delta_h^{r+1} f)(x) \qquad \text{(a.e.)}\end{aligned}$$

which proves the stated assertion.

By Theorem 3.2 it is obvious that $f_{h,r} \in W_X^r[-1, 1]$ and hence $f_{h,r} \in W_X^r[a, b]$.

Now we have to check that $h \in [-\delta_r, \delta_r]$ and $x \in [a, b]$ have the desired property (3.6). Without loss of generality let $h \in [0, \delta_r]$ and $x \in [-1, \delta_r]$. By $[-1, 1/\sqrt{2}] \subset E_h$ and $|k \odot \delta_r| \leqslant k \cdot \delta_r \leqslant 1/\sqrt{2}$ for $k = 1, ..., r-1$ (see (2.5)) we find that $x \in E_h$ (therefore $[a, b] \subset E_h$) and

$$x \oplus h \in [-1, \delta_r \oplus \delta_r] \subset \left[-1, \frac{1}{\sqrt{2}}\right] \subset E_h,$$

$$(x \oplus h) \oplus h \in [-1, 3 \odot \delta_r] \subset \left[-1, \frac{1}{\sqrt{2}}\right] \subset E_h,$$

$$\vdots$$

$$(\cdots(x \oplus \underbrace{h) \oplus \cdots) \oplus h}_{r-2 \text{ summands } h} \in [-1, (r-1) \odot \delta_r] \subset \left[-1, \frac{1}{\sqrt{2}}\right] \subset E_h,$$

thus condition (3.6) is fulfilled. Therefore we have shown that (3.7) holds true for $x \in [a, b]$ (a.e.). ▮

THEOREM 3.4. *Let* $h \in [-1, 1] \setminus \{0\}$ *and* $f \in \mathrm{L}^1_\varphi[-1, 1]$. *Then*

$$\|f - f_h\|_X \leqslant \mathrm{w}_\varphi(f; |h|)_X \qquad \textit{and} \qquad \|Df_h\|_X \leqslant \frac{\mathrm{w}_\varphi(f; |h|)_X}{\arcsin |h|}.$$

Proof. Definition of the Steklov function and Minkowski inequality lead to

$$\begin{aligned} \|f - f_h\|_X &= \left\| \frac{1}{\arcsin h} \int_0^h (f(\bullet) - f(\bullet \oplus t)) \frac{dt}{\sqrt{1-t^2}} \right\|_X \\ &\leqslant \frac{1}{\arcsin |h|} \int_0^{|h|} \|(f(\bullet) - f(\bullet \oplus t))\|_X \frac{dt}{\sqrt{1-t^2}} \\ &\leqslant \frac{1}{\arcsin |h|} \int_0^{|h|} \underbrace{\|(f(\bullet) - f(\bullet \oplus t))\|_X}_{\leqslant \mathrm{w}_\varphi(f; |t|)_X} \frac{dt}{\sqrt{1-t^2}} \\ &\leqslant \mathrm{w}_\varphi(f; |h|)_X \end{aligned} \tag{3.8}$$

and by Theorem 3.2 we obtain

$$\|Df_h\|_X = \frac{\|\Delta_h f\|_X}{\arcsin |h|} \leqslant \frac{\mathrm{w}_\varphi(f; |h|)_X}{\arcsin |h|}. \quad ▮$$

For later use we need the following special notation: For $[c, d] \subset [-1, 1]$ we set

$$\|f\|_{X[c,d]} := \begin{cases} \sup\limits_{x \in [c,d]} |f(x)|, & f \in X = \mathrm{C}[-1, 1] \\ \left(\displaystyle\int_c^d |f(x)|^p \frac{dx}{\sqrt{1-x^2}} \right)^{1/p}, & f \in X = \mathrm{L}^p_\varphi[-1, 1] \end{cases}.$$

THEOREM 3.5. *Let* $r \in \mathbb{N}$, $r \geqslant 2$, $\delta^* := 1/((r^2-1)\sqrt{2})$ *and* $f \in X$. *Moreover, let* $h \in [-\delta^*, \delta^*] \backslash \{0\}$ *and*

$$[c, d] := \begin{cases} [-1, \delta^*]; & h > 0 \\ [-\delta^*, 1]; & h < 0 \end{cases}.$$

Then the function $F_{h,r}$ *defined by*

$$F_{h,r} : [-1, 1] \to \mathbb{R}, \qquad F_{h,r} := \sum_{n=1}^{r} \binom{r}{n} (-1)^{n-1} f_{n \odot h, r},$$

has the following properties

$$\text{(0)} \quad F_{h,r} \in W_X^r[c, d],$$

$$\text{(i)} \quad \|f - F_{h,r}\|_{X[c,d]} \leqslant 2^{1/p} r^r \mathrm{w}_\varphi^r(f; |h|)_X,$$

$$\text{(ii)} \quad \|D^r F_{h,r}\|_{X[c,d]} \leqslant 2^{1/p}(2^r - 1) \frac{\mathrm{w}_\varphi^r(f; |h|)_X}{(\arcsin |h|)^r}.$$

Proof. Let δ_r and $[a, b]$ be defined as in Theorem 3.3. Because of $\delta^* \leqslant \delta_r$ we have $[c, d] \subset [a, b]$ and due to $|n \odot h| \leqslant n \cdot \delta^* \leqslant \delta_r$ for $n = 1, ..., r$ we have $n \odot h \in [-\delta_r, \delta_r]$ and therefore by Theorem 3.3 $f_{n \odot h, r} \in W_X^r[c, d]$ for $n = 1, ..., r$. This yields property (0). Moreover, we have

$$(D^r f_{n \odot h, r})(x) = \frac{(\Delta_{n \odot h}^r f)(x)}{(\arcsin(n \odot h))^r}, \qquad x \in [c, d] \quad \text{(a.e.)}. \tag{3.9}$$

Taking into account that (see (2.3))

$$\|(\Delta_h^r f)(\bullet \oplus t)\|_{X[c,d]} \leqslant 2^{1/p} \|\Delta_h^r f\|_X, \qquad t \in [-1, 1],$$

we find by applying Theorem 2.4

$$\|\Delta_{n \odot h}^r f\|_{X[c,d]} \leqslant 2^{1/p} \sum_{k_1=0}^{n-1} \cdots \sum_{k_r=0}^{n-1} \|\Delta_h^r f\|_X \leqslant 2^{1/p} \cdot n^r \cdot \mathrm{w}_\varphi^r(f; |h|)_X, \tag{3.10}$$

which leads in combination with (3.9) and $\arcsin(n \odot h) = n \cdot \arcsin h$ (see Theorem 2.5) to

$$\|D^r f_{n \odot h, r}\|_{X[c,d]} \leqslant 2^{1/p} \frac{\mathrm{w}_\varphi^r(f; |h|)_X}{(\arcsin |h|)^r}. \tag{3.11}$$

This yields the stated property (ii):

$$\|D^r F_{h,r}\|_{X[c,d]} \leqslant \sum_{n=1}^{r} \binom{r}{n} \|D^r f_{n \odot h, r}\|_{X[c,d]} \leqslant 2^{1/p}(2^r - 1) \frac{\mathrm{w}_\varphi^r(f; |h|)_X}{(\arcsin |h|)^r}.$$

Now we are going to establish part (i). Because $1/((r^2-1)\sqrt{2}) \leqslant 1/(n\sqrt{2})$ for $n=1, ..., r$, the inclusion $[-\delta^*, \delta^*] \subset [-1/(n\sqrt{2}), 1/(n\sqrt{2})]$ holds true. In view of Theorem 2.5 this implies that

$$\begin{aligned}
f_{n\odot h,r}(x) &= \frac{1}{(\arcsin(n\odot h))^r} \int_0^{n\odot h} \cdots \int_0^{n\odot h} f(x\oplus[t_r\oplus\cdots\oplus t_1]) \\
&\quad \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}} \\
&= \frac{1}{(\arcsin h)^r} \int_0^h \cdots \int_0^h f(x\oplus[(n\odot t_r)\oplus\cdots\oplus(n\odot t_1)]) \\
&\quad \times \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}} \\
&= \frac{1}{(\arcsin h)^r} \int_0^h \cdots \int_0^h f(x\oplus[n\odot(t_r\oplus\cdots\oplus t_1)]) \\
&\quad \times \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}}
\end{aligned}$$

for $h\in[-\delta^*, \delta^*]$ and each $x\in[c, d]$. We have used the equality

$$(n\odot t_r)\oplus\cdots\oplus(n\odot t_1) = n\odot(t_r\oplus\cdots\oplus t_1) \tag{3.12}$$

which is true for $t_1, ..., t_r \in [-\delta^*, \delta^*]$ because the left-hand side of (3.12) can be rewritten as a sum with $n\cdot r$ summands and because $t_1, ..., t_r \in [-1/((r^2-1)\sqrt{2}), 1/((r^2-1)\sqrt{2})] \subset [-1/((nr-2)\sqrt{2}), 1/((nr-2)\sqrt{2})]$ implies that the associative law of $n\cdot r$ summands is fulfilled (see remarks concerning (2.1) and (2.2)).

Hence the estimation

$$\begin{aligned}
&f(x) - \sum_{n=1}^{r} \binom{r}{n} (-1)^{n-1} f_{n\odot h,r}(x) \\
&= \frac{(-1)^r}{(\arcsin h)^r} \int_0^h \cdots \int_0^h \sum_{n=0}^{r} \binom{r}{n} (-1)^{r-n} f(x\oplus[n\odot(t_r\oplus\cdots\oplus t_1)]) \\
&\quad \times \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}} \\
&= \frac{(-1)^r}{(\arcsin h)^r} \int_0^h \cdots \int_0^h (\Delta^r_{t_r\oplus\cdots\oplus t_1} f)(x) \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}}
\end{aligned}$$

yields (as in (3.8))

$$\|f-F_{h,r}\|_{X[c,d]}\leqslant \sup_{|\hat{h}|\leqslant|h|} \|\Delta^r_{n\odot\hat{h}} f\|_{X[c,d]}.$$

Combination with inequality (3.10) gives inequality (i). ▮

COROLLARY 3.6. *The conditions of Theorem* 3.5 *may hold. Then the inequalities*

$$\text{(i)}\quad \left.\begin{array}{l}\|f-F_{\delta,r}\|_{X[-1,\delta^*]}\\ \|f-F_{-\delta,r}\|_{X[0,1]}\end{array}\right\}\leqslant 2\cdot 2^{1/p} r^r \mathrm{w}^r_\varphi(f;\delta)_X,$$

$$\text{(ii)}\quad \left.\begin{array}{l}\|D^r F_{\delta,r}\|_{X[-1,\delta^*]}\\ \|D^r F_{-\delta,r}\|_{X[0,1]}\end{array}\right\}\leqslant 2^{1/p}(2^r-1)\,\frac{\mathrm{w}^r_\varphi(f;\delta)_X}{(\arcsin\delta)^r}$$

hold true for all $\delta\in[0,\delta^*]$.

Finally we give an estimation of the modulus for smooth functions. It is a result of Theorem 3.2 and in contrast to the preceding proofs we do not need the associative law.

THEOREM 3.7. *For* $f\in W^r_X[-1,1]$, $r\in\mathbb{N}$ *and* $\delta\in[0,1]$ *we have*:

(i) $\mathrm{w}^r_\varphi(f;\delta)_X\leqslant(2^{1/p})^r\cdot(\arcsin\delta)^r\,\|D^r f\|_X$,

(ii) *if r is even then*

$$\mathrm{w}^{r+s}_\varphi(f;\delta)_X\leqslant(2^{1/p})^r\cdot(\arcsin\delta)^r\,\mathrm{w}^s_\varphi(D^r f;\delta)_X$$

for each $s\in\mathbb{N}$.

Proof. We shall use some basic properties of the differential operator D. With (3.3) and (3.4), $j=1$, we can write

$$D(f\circ\tau_h)=\mu_h\cdot(Df)\circ\tau_h. \tag{3.13}$$

The following can be concluded from Theorem 3.2 and Definition 3.1:

$$(\Delta_h f)(x)=\int_0^h \varphi(x)\,\frac{d}{dx}\,f(x\oplus t)\,\frac{dt}{\sqrt{1-t^2}}=\int_0^h (D(f\circ\tau_t))(x)\,\frac{dt}{\sqrt{1-t^2}}.$$

In abridged notation with (3.13) we obtain

$$\Delta_h f=\int_0^h \mu_t\cdot(Df)\circ\tau_t\,\frac{dt}{\sqrt{1-t^2}}.$$

By iterating this formula we have

$$\Delta_h^r f = \int_0^h \cdots \int_0^h \sigma_r \cdot (D^r f) \circ \tau_{t_1} \circ \cdots \circ \tau_{t_r} \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}}$$

for each $f \in W_X^r[-1, 1]$, where $\sigma_r : [-1, 1] \to \{\pm 1\}$ is a sign function. Using the Minkowski inequality together with (2.3) it follows that

$$\begin{aligned}\|\Delta_h^r f\|_X &\leqslant \int_0^{|h|} \cdots \int_0^{|h|} \|(D^r f) \circ \tau_{t_1} \circ \cdots \circ \tau_{t_r}\|_X \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}} \\ &\leqslant (2^{1/p})^r \cdot \int_0^{|h|} \cdots \int_0^{|h|} \|D^r f\|_X \frac{dt_1}{\sqrt{1-t_1^2}} \cdots \frac{dt_r}{\sqrt{1-t_r^2}} \\ &\leqslant (2^{1/p})^r \cdot (\arcsin |h|)^r \, \|D^r f\|_X. \end{aligned} \tag{3.14}$$

Taking the supremum over h on both sides we find assertion (i).

A direct consequence of Eq. (3.4) is $D^j \circ \Delta_h = \Delta_h \circ D^j$ if j is even. Then $D^r \circ \Delta_h^s = \Delta_h^s \circ D^r$ holds true for every $s \in \mathbb{N}$ and every even r. In combination with (3.14) we obtain

$$\begin{aligned}\|\Delta_h^{r+s} f\|_X &= \|\Delta_h^r(\Delta_h^s f)\|_X \\ &\leqslant (2^{1/p})^r \cdot (\arcsin |h|)^r \, \|D^r(\Delta_h^s f)\|_X \\ &= (2^{1/p})^r \cdot (\arcsin |h|)^r \, \|\Delta_h^s(D^r f)\|_X, \end{aligned}$$

which yields assertion (ii). ∎

4. K-FUNCTIONAL

The K-functional provides an alternative way to characterize the smoothness of functions in place of the moduli of smoothness above.

DEFINITION 4.1. *For $f \in X$ and $\delta \geqslant 0$ the rth order K-functional is defined by*

$$\mathrm{K}_\varphi^r(f; \delta^r)_X := \inf\{\|f-g\|_X + \delta^r \, \|D^r g\|_X \mid g \in W_X^r[-1, 1]\}. \tag{4.1}$$

We show that this K-functional and the modulus of smoothness $\mathrm{w}_\varphi^r(f; \delta)_X$ are equivalent. A lemma is needed first.

LEMMA 4.2. *Let* $f, g \in W_X^r[-1, 1]$, $r \in \mathbb{N}$ *and* $\delta^* \in (0, 1)$. *Then*

(i) $D^r(f \cdot g) = \sum_{k=0}^{r} \binom{r}{k} D^k f D^{r-k} g.$

(ii) $\|D^k f\|_{X[0, \delta^*]} \leqslant M_r \left(\frac{\|f\|_{X[0, \delta^*]}}{(\delta^*)^k} + (\delta^*)^{r-k} \|D^r f\|_{X[0, \delta^*]} \right).$

for $k = 0, ..., r$, *where* $M_r \geqslant 1$ *only depends on* r.

(iii) *There exists a function* χ: $[-1, 1] \to [0, 1]$, $\chi \in W_X^\infty[-1, 1]$ *with*

$$\chi(x) = \begin{cases} 1, & x \in [-1, 0] \\ 0, & x \in [\delta^*, 1] \end{cases} \quad \text{and} \quad \|D^k \chi\|_{C[0, \delta^*]} \leqslant \frac{C_k}{(\delta^*)^k}, \quad k \in \mathbb{N}_0,$$

where $C_k := \|q^{(k)}\|_{C[0, 1]}/q(1)$ *and* $q(t) := \int_0^t e^{1/(u(u-1))}\, du$, $t \in [0, 1]$.

Proof. Let $f \in W_X^r[-1, 1]$. By substitution $x = \cos t$ for $t \in [0, \pi]$ and the relation

$$(Df)(\cos t) = \sqrt{1 - \cos^2 t}\, f'(\cos t) = \sin t\, f'(\cos t) = -f(\cos t)' \qquad \text{(a.e.)}$$

one finds that

$$(D^k f)(\cos t) = (-1)^k f(\cos t)^{(k)} \qquad \text{(a.e.)} \tag{4.2}$$

holds true for $k \in \mathbb{N}_0$.

(i) By using the Leibniz formula and (4.2) we have

$$\begin{aligned} (D^r(f \cdot g))(\cos t) &= (-1)^r (f(\cos t) \cdot g(\cos t))^{(r)} \\ &= (-1)^r \sum_{k=0}^{r} \binom{r}{k} f(\cos t)^{(k)} \cdot g(\cos t)^{(r-k)} \\ &= \sum_{k=0}^{r} \binom{r}{k} (D^k f)(\cos t) \cdot (D^{(r-k)} g)(\cos t) \end{aligned}$$

for $t \in [0, \pi]$ (a.e.). $x = \cos t$ yields the statement (i).

(ii) Let $f \in W_X^r[0, \pi]$ and $g(t) := f(\cos t)$ for $t \in [\arccos \delta^*, (\pi/2)]$. Then

$$g, g^{(r)} \in \begin{cases} \mathrm{C}\left[\arccos \delta^*, \frac{\pi}{2}\right], & p = \infty \\ \mathrm{L}^p\left[\arccos \delta^*, \frac{\pi}{2}\right], & 1 \leqslant p < \infty \end{cases}$$

and by using Lemma 2.1 in [4] with $[a, b] := [\arccos \delta^*, (\pi/2)]$ and $\delta^* \leqslant (\pi/2) - \arccos \delta^* \leqslant (\pi/2)\, \delta^*$ we get

$$\| g^{(k)} \|_p \leqslant M(r, k) \cdot \left\{ \frac{\| g \|_p}{((\pi/2) - \arccos \delta^*)^k} + ((\pi/2) - \arccos \delta^*)^{r-k} \, \| g^{(r)} \|_p \right\}$$

$$\leqslant M(r, k) \cdot \left\{ \frac{\| g \|_p}{(\delta^*)^k} + (\delta^*)^{r-k} \, \| g^{(r)} \|_p \right\}$$

for $k = 0, ..., r$ and $M(r, k) \geqslant 1$, where $\| \bullet \|_p$ denotes the unweighted L^p norm. Because of

$$\| g \|_p^p = \int_{\arccos \delta^*}^{\pi/2} |f(\cos t)|^p \, dt = -\int_{\delta^*}^{0} |f(x)|^p \frac{dx}{\sqrt{1 - x^2}} = \| f \|_{X[0, \delta^*]}^p,$$

and by $M_r := \max_{k = 0, ..., r} M(r, k)$ and using (4.2) we find assertion (ii).

(iii) Let

$$q_0(t) := \begin{cases} e^{1/(t(t-1))}, & t \in [0, 1] \\ 0, & t \in \mathbb{R} \setminus [0, 1] \end{cases}.$$

Then $q_0 \in C^\infty(\mathbb{R})$. By setting

$$q_1(t) := \left(\int_0^1 q_0(u) \, du \right)^{-1} \cdot \int_{-\infty}^{t} q_0(u) \, du$$

we have $q_1 \in C^\infty(\mathbb{R})$ and

$$q_1(t) = \begin{cases} 0, & t \leqslant 0 \\ 1, & t \geqslant 1 \end{cases}.$$

Thus

$$q_2(t) := q_1 \left(\frac{t - \arccos \delta^*}{(\pi/2) - \arccos \delta^*} \right)$$

fulfills $q_2 \in C^\infty[0, \pi]$ and

$$q_2(t) = \begin{cases} 0, & t \in [0, \arccos \delta^*] \\ 1, & t \in [(\pi/2), \pi] \end{cases}.$$

Finally, let $\chi(x) := q_2(\arccos x)$, $x \in [-1, 1]$. Then

$$\chi(x) = \begin{cases} 1, & x \in [-1, 0] \\ 0, & x \in [\delta^*, 1] \end{cases}$$

and the inequality $\delta^* \leqslant (\pi/2) - \arccos \delta^*$ leads to

$$\begin{aligned}\|D^k\chi\|_{C[0,\,\delta^*]} &= \|(\chi\circ\cos)^{(k)}\|_{C[\arccos\delta^*,\,(\pi/2)]} \leqslant \|q_2^{(k)}\|_{C[\arccos\delta^*,\,(\pi/2)]}\\ &\leqslant (\delta^*)^{-k}\left\|q_1^{(k)}\left(\frac{\bullet - \arccos\delta^*}{(\pi/2)-\arccos\delta^*}\right)\right\|_{C[\arccos\delta^*,\,(\pi/2)]}\\ &\leqslant (\delta^*)^{-k}\,\|q_1^{(k)}\|_{C[0,\,1]}\\ &= (\delta^*)^{-k}\,\|q^{(k)}\|_{C[0,\,1]}/q(1)\end{aligned}$$

since $q_1(t) = q(t)/q(1)$ for $t \in [0, 1]$. Obviously $\chi \in W^\infty_{C[-1,\,1]} \subset W^\infty_{L^p_\varphi[-1,\,1]}$. This completes the proof. ∎

Remark 4.3. An immediate consequence of the proof of [4, Lemma 2.1] is that we can choose $M_2 = 8$ in Lemma 4.2(ii).

In Lemma 4.2(iii) we have obviously $C_0 = 1$. By using $q(1) = 142.25\ldots$ an elementary compuation yields that the inequalities $C_1 \leqslant 3$ and $C_2 \leqslant 12$ hold true.

Now we are ready to state the main result.

THEOREM 4.4. *Let $r \in \mathbb{N}$ and*

$$\delta^* := \begin{cases} 1 & \textit{for} \quad r = 1\\ \dfrac{1}{(r^2-1)\sqrt{2}} & \textit{for} \quad r \geqslant 2.\end{cases}$$

For each $f \in X$ and $\delta \in [0, \delta^*]$ the estimate

$$\frac{1}{(2\cdot 2^{1/p})^r}\,\mathrm{w}^r_\varphi(f;\delta)_X \leqslant \mathrm{K}^r_\varphi(f;\delta^r)_X \leqslant c_r\mathrm{w}^r_\varphi(f;\delta)_X$$

holds true where

$$c_r = \begin{cases} 2, & r = 1\\ 2a + 3b + 2M_r(a+b)\displaystyle\sum_{j=0}^{r}\binom{r}{j}C_j, & r \geqslant 2\end{cases}$$

and $a := 2^{1/p}r^r$, $b := 2^{1/p}(2^r - 1)$ and C_j, M_r are defined in Lemma 4.2.

Proof. Let $g \in W^r_X[-1, 1]$, $r \in \mathbb{N}$, be chosen arbitrarily. By using (2.4), Theorem 3.7(i) and $\arcsin \delta \leqslant (\pi/2)\delta$ for $\delta \in [0, 1]$ we obtain

$$\begin{aligned} \mathrm{w}_\varphi^r(f;\delta)_X &\leqslant \mathrm{w}_\varphi^r(f-g;\delta)_X+\mathrm{w}_\varphi^r(g;\delta)_X \\ &\leqslant (2\cdot 2^{1/p})^r\cdot\|f-g\|_X+(2^{1/p})^r\cdot\left(\frac{\pi}{2}\delta\right)^r\cdot\|D^rg\|_X \\ &\leqslant (2\cdot 2^{1/p})^r\cdot\{\|f-g\|_X+\delta^r\,\|D^rg\|_X\} \end{aligned}$$

for all $g\in W_X^r[-1,1]$. This yields the lower inequality of the assertion.

For $r=1$ the upper inequality can be verified simply by applying Theorem 3.4:

$$\mathrm{K}_\varphi^1(f;\delta)_X\leqslant\|f-f_\delta\|_X+\arcsin\delta\,\|Df_\delta\|_X\leqslant 2\mathrm{w}_\varphi^1(f;\delta)_p.$$

Let $r\geqslant 2$ and $\delta\in(0,\delta^*]$. For the upper inequality we shall use the Steklov functions $F_{\delta,r}$ and $F_{-\delta,r}$ of Theorem 3.5. It may be pointed out that the restriction of $F_{\delta,r}$ on the subinterval $[-1,0]$ and $F_{-\delta,r}$ on $[0,1]$ belong to the Sobolev spaces $W_X^r[-1,0]$ and $W_X^r[0,1]$ respectively. Consequently it is our task to construct a function $\tilde f_{\delta,r}\in W_X^r[-1,1]$ which has the desired smoothness properties on the whole interval $[-1,1]$.

For abbreviation let us set $f_1:=F_{\delta,r}, f_2:=F_{-\delta,r}$. Then $f_1\in W_X^r[-1,\delta^*]$, $f_2\in W_X^r[0,1]$ and

$$\begin{aligned} \|f-f_1\|_{X[-1,\delta^*]}&\leqslant a\cdot\mathrm{w}_\varphi^r(f;\delta)_X, &\qquad \|D^rf_1\|_{X[-1,\delta^*]}&\leqslant b\cdot\delta^{-r}\mathrm{w}_\varphi^r(f;\delta)_X, \\ \|f-f_2\|_{X[0,1]}&\leqslant a\cdot\mathrm{w}_\varphi^r(f;\delta)_X, &\qquad \|D^rf_2\|_{X[0,1]}&\leqslant b\cdot\delta^{-r}\mathrm{w}_\varphi^r(f;\delta)_X, \end{aligned}$$

where $a=2^{1/p}r^r$ and $b=2^{1/p}(2^r-1)$ (see Corollary 3.6).

Now let $\tilde f:=\chi f_1+(1-\chi)f_2=(f_1-f_2)\chi+f_2$, where χ is defined in Lemma 4.2(iii). Then $\tilde f\in W_X^r[-1,1]$ and

$$\tilde f(x)=\begin{cases} f_1(x), & x\in[-1,0] \\ f_2(x), & x\in[\delta^*,1] \end{cases}.$$

It must be shown that $\tilde f$ has the desired properties of smoothness on $[-1,1]$. Because of $\|\chi\|_\infty\leqslant 1$ we have

$$\begin{aligned} \|f-\tilde f\|_X &\leqslant \|(f-f_1)\chi\|_X+\|(f-f_2)(1-\chi)\|_X \\ &\leqslant \|f-f_1\|_{X[-1,\delta^*]}+\|f-f_2\|_{X[0,1]} \\ &\leqslant 2a\cdot\mathrm{w}_\varphi^r(f;\delta)_X. \end{aligned} \tag{4.3}$$

On the other hand, the following estimate holds true:

$$\begin{aligned} \|D^r\tilde f\|_X &\leqslant \|D^r\tilde f\|_{X[-1,0]}+\|D^r\tilde f\|_{X[0,\delta^*]}+\|D^r\tilde f\|_{X[\delta^*,1]} \\ &\leqslant \|D^rf_1\|_{X[-1,0]}+\|D^r\tilde f\|_{X[0,\delta^*]}+\|D^rf_2\|_{X[\delta^*,1]} \\ &\leqslant 2b\cdot\delta^{-r}\mathrm{w}_\varphi^r(f;\delta)_X+\|D^r\tilde f\|_{X[0,\delta^*]}. \end{aligned} \tag{4.4}$$

Using Lemma 4.2(i), (ii) and (iii) we obtain

$$\begin{aligned}
\|D^r\tilde{f}\|_{X[0,\delta^*]} &\leqslant \sum_{j=0}^{r} \binom{r}{j} \|D^j(f_1-f_2)\, D^{r-j}\chi\|_{X[0,\delta^*]} + \|D^r f_2\|_{X[0,\delta^*]} \\
&\leqslant \sum_{j=0}^{r} \binom{r}{j} \|D^j(f_1-f_2)\|_{X[0,\delta^*]}\, \|D^{r-j}\chi\|_{C[0,\delta^*]} \\
&\quad + \|D^r f_2\|_{X[0,\delta^*]} \\
&\leqslant M_r \sum_{j=0}^{r} \binom{r}{j} C_{r-j} \left\{ \frac{\|f_1-f_2\|_{X[0,\delta^*]}}{(\delta^*)^r} + \|D^r(f_1-f_2)_{X[0,\delta^*]} \right\} \\
&\quad + \|D^r f_2\|_{X[0,\delta^*]} \\
&\leqslant M_r \sum_{j=0}^{r} \binom{r}{j} C_{r-j} \left\{ \frac{2a\cdot \mathrm{w}_\varphi^r(f;\delta)_X}{(\delta^*)^r} + \frac{2b\cdot \mathrm{w}_\varphi^r(f;\delta)_X}{(\delta^*)^r} \right\} \\
&\quad + \frac{b\cdot \mathrm{w}_\varphi^r(f;\delta)_X}{(\delta^*)^r} \\
&\leqslant \left(2M_r(a+b) \sum_{j=0}^{r} \binom{r}{j} C_{r-j} + b \right) \frac{\mathrm{w}_\varphi^r(f;\delta)_X}{(\delta^*)^r}
\end{aligned}$$

and in combination with (4.4)

$$\|D^r\tilde{f}\|_X \leqslant \left(2M_r(a+b) \sum_{j=0}^{r} \binom{r}{j} C_j + 3b \right) \frac{\mathrm{w}_\varphi^r(f;\delta)_X}{\delta^r}. \tag{4.5}$$

Now we return to the proof of the upper inequality of the assertion for $r \geqslant 2$. Taking into account (4.3) and (4.5) we obtain

$$\begin{aligned}
K_\varphi^r(f;\delta)_X &\leqslant \|f-\tilde{f}\|_X + \delta^r\, \|D^r\tilde{f}\|_X \\
&\leqslant \left(2a + 2M_r(a+b) \sum_{j=0}^{r} \binom{r}{j} C_j + 3b \right) \mathrm{w}_\varphi^r(f;\delta)_X. \quad \blacksquare
\end{aligned}$$

Remark 4.5. Concerning the proof we have also shown that there exist Steklov functions $\tilde{f}_{\delta,r} := \tilde{f}$ for $\delta \in (0, 1/((r^2-1)\sqrt{2})]$ which have the approximation property (4.3) and the smoothness property (4.5) on the whole interval $[-1, 1]$. This result is an extension of Theorem 3.5.

By making use of Remark 4.3 we get for the second order the following constants.

COROLLARY 4.6. *Let $f \in X$.*

(i) *For each $\delta \in [0, 1/(3\sqrt{2})]$ the following estimate holds true*

$$\frac{1}{4 \cdot 4^{1/p}} \mathrm{w}_\varphi^2(f; \delta)_X \leqslant \mathrm{K}_\varphi^2(f; \delta^2)_X \leqslant 2145 \cdot 2^{1/p} \mathrm{w}_\varphi^2(f; \delta)_X.$$

(ii) *For each $\delta \in (0, 1/(3\sqrt{2})]$ there exists a Steklov function $\tilde{f}_{\delta, 2} \in W_X^2[-1, 1]$ so that*

$$\|f - \tilde{f}_{\delta, 2}\|_X \leqslant 8 \cdot 2^{1/p} \mathrm{w}_\varphi^2(f; \delta)_X \quad \text{and} \quad \|D^2 \tilde{f}_{\delta, 2}\|_X \leqslant 2137 \cdot 2^{1/p} \frac{\mathrm{w}_\varphi^2(f; \delta)_X}{\delta^2}.$$

5. APPLICATIONS AND OTHER MODULI

Having proved the equivalence between the modulus and the K-functional, we are now able to establish the fundamental theorem of best algebraic approximation. The next theorem characterizes the behaviour of the best algebraic approximation by means of the modulus of smoothness w_φ^r.

Let us set $\mathrm{E}_n(f)_X = \inf_{p_n \in \mathbb{P}_n} \|f - p_n\|_X$ where $\mathbb{P}_n$ denotes the set of all algebraic polynomials of degree n.

THEOREM 5.1. *Let $f \in X$, $r \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $0 < \alpha < r$. Then the following equivalence holds true*:

$$\mathrm{E}_n(f)_X = O(n^{-\alpha})(n \to \infty) \Leftrightarrow \mathrm{w}_\varphi^r(f; \delta)_X = O(\delta^\alpha)(\delta \to 0).$$

Proof. The left hand side is valid iff

$$\mathrm{E}_n(f \circ \cos)_p = O(n^{-\alpha}) \qquad (n \to \infty) \tag{5.1}$$

where $\mathrm{E}_n(f \circ \cos)_p = \inf_{t_n \in \Pi_n} \|f \circ \cos - t_n\|_p$, $\|g\|_p := (\int_0^{2\pi} |g(t)|^p \, dt)^{1/p}$, $p \in [1, \infty)$, $(\|g\|_\infty := \sup_{t \in [0, 2\pi]} |g(t)|)$ for 2π periodic functions $g: \mathbb{R} \to \mathbb{R}$ and Π_n is the set of all trigonometric polynomials of degree n. It is well known that (5.1) is equivalent to

$$\inf\{\|f \circ \cos - \tilde{g}\|_p + \delta^r \|\tilde{g}^{(r)}\|_p \mid \tilde{g}\, 2\pi \text{ periodic}, \tilde{g}^{(r)} \text{ exists}\} = O(\delta^\alpha) \tag{5.2}$$

for $\delta \to 0$. Concerning (4.2) a short consideration yields that the left hand side of (5.2) is equal to

$$\inf\{\|f \circ \cos - g \circ \cos\|_p + \delta^r \|(g \circ \cos)^{(r)}\|_p \mid g \in W_X^r[-1, 1]\}$$
$$= K_\varphi^r(f; \delta^r)_X. \tag{5.3}$$

Theorem 4.4 completes the proof. ∎

The connecting link between the modulus and $E_n(f)_X$ in the above proof is the K-functional. With the aid of the K-functional we can also compare the algebraic modulus with the Butzer–Stens modulus.

In [1] Butzer and Stens defined the so-called Chebyshev translation operator for $f \in X$ by

$$(T_h f)(x) := \tfrac{1}{2}\{f(xh + \sqrt{1-x^2}\sqrt{1-h^2}) + f(xh - \sqrt{1-x^2}\sqrt{1-h^2})\},$$
$$x, h \in [-1, 1].$$

In terms of the difference operator

$$\bar{\Delta}_h f := T_h f - f, \qquad \bar{\Delta}_h^r := \bar{\Delta}_h \circ \cdots \circ \bar{\Delta}_h, \qquad r \text{ times},$$

the Butzer–Stens modulus of smoothness for $f \in X$ is then given by

$$\omega_T^r(f; \eta) = \sup_{\eta \leqslant h \leqslant 1} \|\bar{\Delta}_h^r f\|_X, \qquad \eta \in [0, 1].$$

We now show that our modulus $\mathrm{w}_\varphi^{2r}(f; \delta)_X$ of even order and the Butzer–Stens modulus $\omega_T^r(f; \cos\delta)$ are equivalent.

THEOREM 5.2. *Let $r \in \mathbb{N}$. For $f \in X$ and $\delta \in [0, 1/((4r^2-1)\sqrt{2})]$ there exist constants $c_r, C_r > 0$ depending only on r so that*

$$c_k \omega_T^r(f; \cos\delta) \leqslant \mathrm{w}_\varphi^{2r}(f; \delta)_X \leqslant C_k \omega_T^r(f; \cos\delta).$$

Proof. For defining a corresponding K-functional of the Butzer–Stens modulus we need a differential operator D_T. In [1] it is defined by a strong derivative by means of the Chebyshev translation operator. In [2, Theorem 6] it is shown that D_T has the representation

$$(D_T f)(x) = (1-x^2) f''(x) - x f'(x), \qquad x \in [-1, 1] \quad \text{(a.e.)} \qquad (5.4)$$

iff the right hand side exists. As usual the power is defined by $D_T^k := D_T \circ \cdots \circ D_T$ k times. Then the K-functional is defined as

$$K_T(f; \delta^{2r}) := \inf\{\|f-g\|_X + \delta^{2r}\|D_T^r g\|_X \mid D_T^r g \text{ exists in } X\}, \qquad \delta \geqslant 0.$$

The equivalence between this K-functional and $\omega_T^r(f; \cos\delta)$ has been given by Butzer and Stens in [1]. Observing that $D_T f = D^2 f$ holds true because of (5.4), one has $K_T(f; \delta^{2r}) = K_\varphi^{2r}(f; \delta^{2r})_X$. Because of the equivalence of $K_\varphi^{2r}(f; \delta^{2r})_X$ and $\mathrm{w}_\varphi^{2r}(f; \delta)_X$ for $\delta \in [0, 1/((4r^2-1)\sqrt{2})]$ (see Theorem 4.4) we have established that $\mathrm{w}_\varphi^{2r}(f; \delta)_X$ and $\omega_T^r(f; \cos\delta)$ are equivalent. ∎

There are other moduli of smoothness which are suitable for characterizing best algebraic approximation. In their book [5] Ditzian and Totik measured the smoothness of functions in weighted L^p spaces on $[-1, 1]$ in terms of the main part modulus of smoothness. For $f \in L^p_\varphi[-1, 1]$ the main part modulus is given by

$$\Omega^r_\varphi(f, \delta)_X = \sup_{|h| \leqslant \delta} \|\hat{\Delta}^r_{h\varphi} f\|_{L^p_\varphi[-1+2r^2h^2, 1-2r^2h^2]}, \qquad \varphi(x) = \sqrt{1-x^2},$$

where $(\hat{\Delta}^r_h f)(x) = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f(x + r(h/2) - kh)$ if $x, x + r(h/2) - kh \in [-1, 1]$ and $(\hat{\Delta}^r_h f)(x) = 0$ otherwise. Using suitably defined K-functionals they were able to prove that $\mathrm{E}_n(f)_X = O(n^{-\alpha})$ is equivalent to $\Omega^r_\varphi(f; \delta)_X = O(\delta^\alpha)$ $(0 < \alpha < r)$. For the unweighted space $C[-1, 1]$ the main part modulus Ω^r_φ can be replaced by the Ditzian–Totik modulus

$$\omega^r_\varphi(f; \delta)_\infty = \sup_{|h| \leqslant \delta} \|\hat{\Delta}^r_{h\varphi} f\|_\infty$$

which is equivalent to the specific K-functional

$$\hat{\mathrm{K}}^r_\varphi(f; \delta^r)_\infty = \inf\{ \|f - g\|_\infty + \delta^r \|\varphi^r g^{(r)}\|_\infty \mid g^{(r-1)} \in \mathrm{AC}[-1, 1]\}.$$

Because of $D^1 f = \varphi f'$ and Theorem 4.4 the moduli of first order w^1_φ and ω^1_φ are equivalent. This does not hold true for higher orders $r \geqslant 2$. Let $f \in \mathbb{P}_{r-1} \backslash \mathbb{P}_0$ then $\hat{\mathrm{K}}^r_\varphi(f; \delta^r)_\infty = 0$ (because $\varphi^r g^{(r)} = 0$ for $g = f$). But with the aid of (5.3) and the fact that $(g \circ \cos)^{(r)} \neq 0$ for $g = f$ one finds $\mathrm{K}^r_\varphi(f; \delta^r)_{C[-1,1]} \neq 0$. Therefore w^r_φ and ω^r_φ are not equivalent for $r \geqslant 2$ on $C[-1, 1]$. Consequently w^r_φ, $r \geqslant 2$, is also not equivalent to the τ-modulus ([9]) which is equivalent to $\omega^r_\varphi(f; \delta)_\infty$ on $X = \mathrm{C}[-1, 1]$. The question is still open as to whether w^r_φ is equivalent to Ω^r_φ on the weighted spaces $X = L^p_\varphi[-1, 1]$.

Finally, it should also be pointed out that for unweighted spaces $L^p[-1, 1]$ no comparison is possible because w^r_φ is not well-defined on $L^p[-1, 1]$ which can be seen as follows.

Lemma 5.3. *Let $p \in [1, \infty)$ and $h \in [-1, 1] \backslash \{0\}$. Then*

$$\Delta_h(L^p[-1, 1]) \not\subset L^p[-1, 1].$$

Proof. Without loss of generality we can assume $h > 0$. Let us define the function

$$f \colon [-1, 1] \to \mathbb{R}, \qquad f(x) := \left(\frac{1 \oplus (x \ominus h)}{1 \oplus x} \right)^{1/p}.$$

A short calculation yields

$$(f(x))^p = \frac{|\sqrt{1-x^2}\,\sqrt{1-h^2}+xh|}{\sqrt{1-x^2}} = \left|\sqrt{1-h^2} + \frac{xh}{\sqrt{1-x^2}}\right|$$

from which we obtain

$$\begin{aligned}\|f\|_p^p &\leqslant \int_{-1}^{1} \sqrt{1-h^2}\,dx + h\int_{-1}^{1} \frac{|x|}{\sqrt{1-x^2}}\,dx \\ &= 2\sqrt{1-h^2} + 2h\cdot[-\sqrt{1-x^2}]_0^1 = 2\sqrt{1-h^2}+2h < \infty,\end{aligned}$$

i.e. $f \in L^p[-1,1]$. Let us assume that $\Delta_h f \in L^p[-1,1]$. It will be shown that this assumption gives a contradiction. The hypothesis implies that $f(\bullet \oplus h) \in L^p[-1,1]$ and by substitution $x \mapsto x \ominus h$ and $x \mapsto x \oplus h$ respectively we conclude

$$\begin{aligned}\|f(\bullet\oplus h)\|_p^p &= \int_{-1\oplus h}^{1} |f(x)|^p \frac{1\oplus(x\ominus h)}{1\oplus x}\,dx + \underbrace{\int_{1\ominus h}^{1} |f(x)|^p \frac{1\oplus(x\oplus h)}{1\oplus x}\,dx}_{\geqslant 0} \\ &\geqslant \int_0^1 \frac{1\oplus(x\ominus h)}{1\oplus x}\cdot\frac{1\oplus(x\ominus h)}{1\oplus x}\,dx \\ &= \int_0^1 \left((1-h^2) + h\sqrt{1-h^2}\,\frac{2x}{\sqrt{1-x^2}} + h^2\frac{x^2}{1-x^2}\right)dx \\ &= 1-h^2+h\sqrt{1-h^2}+h^2\cdot\left[-x+\frac{1}{2}\log\frac{1+x}{1-x}\right]_0^{1-}.\end{aligned}$$

Because of assumption $h \neq 0$ we have $\|f(\bullet\oplus h)\|_p = \infty$ which is a contradiction to $\Delta_h f \in L^p[-1,1]$. ∎

Lemma 5.3 shows that the difference operator $(\Delta_h f)(x) = f(x\oplus h) - f(x)$ given in terms of the algebraic addition $\oplus$ is not suitable for defining a modulus of smoothness for unweighted L^p spaces on $[-1,1]$. The question is open as to whether there exists an algebraic operation $\hat{\oplus}$ defined on the unit interval $[-1,1]$ which is suitable for measuring smoothness in order to characterize best algebraic approximation.

REFERENCES

1. P. L. Butzer and R. L. Stens, Chebyshev transform methods in the theory of best algebraic approximation, *Abh. Math. Sem. Hamburg* **45** (1976), 165–190.
2. P. L. Butzer and R. L. Stens, The operational properties of the Chebyshev transform. I. General properties, *Funct. Approx. Comment. Math.* **5** (1977), 129–160.

3. R. A. DeVore, $L_p[-1, 1]$ approximation by algebraic polynomials, *in* "Linear Spaces and Approximation" (P. L. Butzer and B. Sz. Nagy, Eds.), pp. 397–406, Birkhäuser, Basel, 1978.
4. Z. Ditzian, On interpolation of $L^p[a, b]$ and weighted Sobolev spaces, *Pacific J. Math.* **90** (1980), 307–323.
5. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, New York, 1987.
6. M. Felten, "Ein neuer Glättemodul für nichtperiodische Funktionen," dissertation. Universität Dortmund, 1992.
7. M. Felten, A modulus of smoothness based on an algebraic addition, *Aequationes Math.*, to appear.
8. K. G. Ivanov, Direct and converse theorems for the best algebraic approximation in $C[-1, 1]$ and $L^p[-1, 1]$, *C. R. Acad. Bulgare Sci.* **33**(10) (1980), 1309–1312.
9. K. G. Ivanov, A characterization of weighted Peetre K-functionals, *J. Approx. Theory* **56** (1989), 185–211.
10. V. P. Motorniĭ, Approximation of functions by algebraic polynomials in L_p metric, *Izv. Akad. Nauk SSSR* **35** (1971), 874–899 [in Russian].
11. A. F. Timan, Strengthening of Jackson theorem on the best approximation of continuous functions on a finite segment of the real axis, *Dokl. Akad. Nauk SSSR* **78** (1951), 17–20 [in Russian].